Weak Stochastic Bisimulation for Non-Markovian Processes*

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Abstract. In this paper we introduce a novel notion of bisimulation to properly capture the behavior of stochastic systems with general distributions. The key idea consists in the identification of different sequences of random variables if the *additions* of the random variables of each sequence are identically distributed. That is, we will not only identify sequences of internal actions with one of them (as it is usually done in weak bisimulations) but we will also reduce (in some conditions) sequences of stochastic transitions to only one transition. Therefore, we will identify processes that are considered non-equivalent in previous notions of bisimulation for this kind of languages.

1 Introduction

Process algebras [Hoa85,Hen88,Mil89,BW90] are a powerful mechanism to specify the functional behavior of concurrent and distributed systems. Nevertheless, the original formulations were not able to accurately represent systems where *quantitative* information, such as time and probabilities, played a fundamental role. Therefore, several timed and/or probabilistic extensions of process algebras have been proposed in the literature. In timed process algebras time information is mainly specified in two different forms: Either by adding a delay operator or by including information about the time when actions are enabled. So, no information about the *probability* associated with this temporal information is included. However, there are processes in which we would like to specify that the probability of performing an action changes as time passes. For example, the event of cars arriving to a petrol station follows a Poisson distribution.

A natural evolution has appeared with the introduction of *stochastic process algebras* (e.g. [GHR93,ABC$^+$94,Hil96,Her98,BG98,HS00,LN01,HHK02]) where time information is incremented with some probabilistic information. For example, one may specify a system where a message is expected to be received with probability $\frac{1}{2}$ in the interval $(0,1]$, with probability $\frac{1}{4}$ in $(1,2]$, and so on. This is an advantage with respect to timed processes where we could only specify that

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the message arrives in the interval $(0, \infty)$. There are two main ways of specifying this (stochastic) time information: Either by adding a *timer* to actions or by specifying (random) delays. In the first case, an expression as $(a, \xi) \,;\, P$ indicates that the probability of performing a, before time t has passed, is equal to the probability that the random variable ξ takes values less than or equal to t. After a is performed, the process behaves as P. For example, if ξ is uniformly distributed over the interval $[1, 2]$ the probability of executing a before time $\frac{2}{3}$ is equal to 0, the probability of executing a before time $\frac{3}{2}$ is equal to $\frac{1}{2}$, and so on. Examples of such languages are [Hil96,BG98,LN00]. In the second case, an expression as $\xi \,;\, P$ indicates that the process will be delayed according to ξ and then it will behave as P. Again, if ξ is uniformly distributed over $[1, 2]$, P starts before time $\frac{5}{4}$ with probability $\frac{1}{4}$, before time 2 with probability 1, etc. Examples of such languages are [Her98,DKB98,LN01,HHK02,BG02]. In this paper we have preferred to take the second approach because the separation between stochastic and functional behaviors makes (semantic) definitions simpler.

With a few exceptions most stochastic models consider that distributions are restricted to be exponential. This restriction simplifies several of the problems that appear when considering general distributions. In particular, some quantities of interest, like reachability probabilities or steady-state probabilities, can be efficiently calculated by using well known methods on Markov chains. Besides, the (operational) definition of the language is usually simpler than the one for languages allowing general distributions. Nevertheless, this restriction does not allow to properly specify some kind of systems where time distributions are not exponential. Moreover, the main weakness of non-exponential models, the analysis of properties, can be (partially) overcome by restricting the class of distributions. A good candidate are phase-type distributions. This kind of distributions has some good properties: they are closed under minimum, maximum, and convolution, and any other distribution over the interval $(0, \infty)$ can be approximated by arbitrarily accurate phase-type distributions. Moreover, the analysis of performance measures can be efficiently done in some general cases.

Our aim in this paper is to define an equivalence relation that, on the one hand, properly captures the branching structure of stochastic processes and that, on the other hand, it is sufficiently abstract with respect to stochastic transitions. Even though most of the semantics presented for stochastic process algebras are bisimulation-like semantics,[1] in our opinion previous definitions do not adequately abstract stochastic transitions because these transitions are (essentially) treated as *visible* ones. For example, consider the processes depicted in Figure 1 whose initial states are s_1 and s_2. We think that s_1 and s_2 could be equivalent (for a suitable choice of ξ in the second process) but they are usually considered to be non-equivalent. The reason is that s_1 may consecutively perform two stochastic transitions while s_2 may perform only one. This situation represents a big difference with respect to timed process algebras where, for example, the following process delay(1) ; delay(2) ; Q is always considered to be equivalent to

[1] Two notable exceptions are [BC00,LN01] where testing semantics are defined for stochastic process algebras.

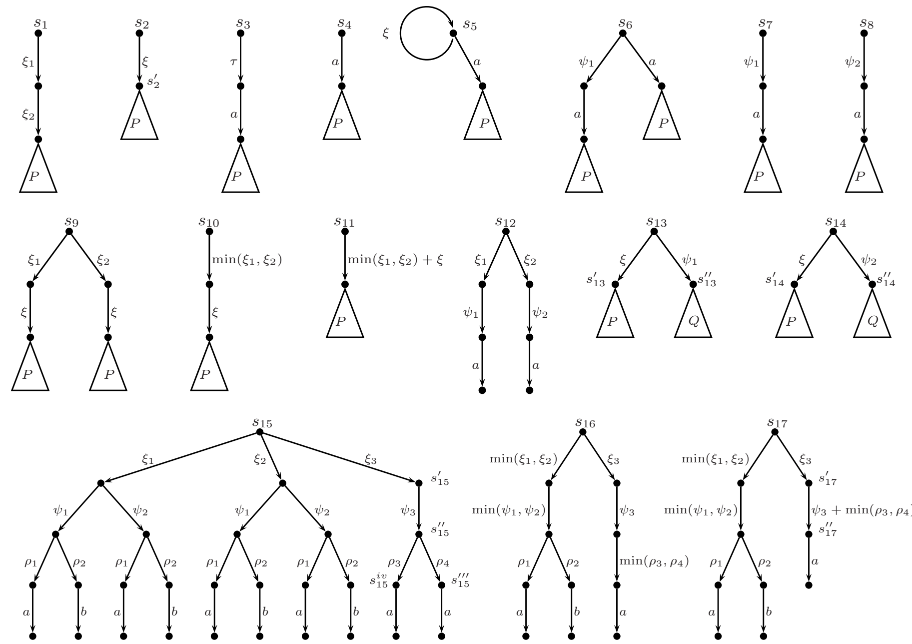

Fig. 1. Examples of stochastic processes.

delay(3) ; Q, where delay(n) indicates that the process is delayed n units of time (in this case, the delay is fixed). Following this reasoning, we should be able to identify s_1 and s_2 for an appropriate ξ. A good candidate is to consider that ξ is distributed as the addition of ξ_1 and ξ_2. But here appears a drawback of exponential distributions: They are not closed under addition, that is, if ξ_1 and ξ_2 are exponentially distributed, then the random variable $\xi_1 + \xi_2$ is not exponentially distributed. For this reason, the restriction to exponential distributions does not allow to define a semantics with our expected characteristics. However, to the best of our knowledge, even when considering general distributions, such a semantics has not been previously presented.

We propose a new bisimulation semantics for stochastic processes. Regarding the treatment of *usual* actions, our definition follows the classical weak bisimulation. Let us present some simple examples to show which processes we would like to identify according to their stochastic behaviors. Consider s_3, s_4 and s_5 in Figure 1. The first two ones are weakly bisimilar, so we would like to identify them. On the contrary, s_5 is always considered to be *different*. The reason is similar to the case of s_1 and s_2: A stochastic transition may be performed from s_5 while this is not the case for s_3 or s_4. But this transition leads to the same state s_5, so the intended meaning of this process is "If a is offered by the environment, then a is performed; if the delay is consumed, then the process comes back to the initial state". In other words, we should not be able to distinguish between s_4 and s_5 because both can perform a as soon as it is offered by the environment. Using a similar reasoning, s_6 should be also equivalent to the previous processes. Consider now s_7 and s_8. Assuming that ψ_1 and ψ_2 are not identically distributed, the probabilities of performing a after n units of time have passed are different. So, s_7 and s_8 should not be identified (and they are usually not). Consider now s_9, s_{10} and s_{11}. The first two processes are usually identified in stochastic bisimulation semantics (so we do). The idea is that the *fastest* delay will be taken, and so we must consider a random variable distributed as the *minimum* of ξ_1 and ξ_2. Besides, following a reasoning similar to that for s_1 and s_2, we also consider that s_{11} is equivalent to both s_9 and s_{10}. This example presents the key point of our semantics: First, we have to check if some transitions may be *joined* (by using the minimum) because they lead to *equivalent* states, and then we have to check if the *continuation* must be added to the resulting random variable. But there are situations where no stochastic transitions are joined at all. For example, if we consider s_{12} and we take into account the previous comments on s_7 and s_8, we should not join the two initial random variables because they lead to non-equivalent states. So, this process will not be equivalent to any other process (up to trivial renaming of random variables).

The combination of *minimums* and *additions*, that we have commented before, produces that the definition of the desired semantics is far from trivial. Consider s_{15}, s_{16}, and s_{17} in Figure 1. All these processes will be identified in our semantics. This example illustrates that the minimum is chosen among all the random variables that lead to equivalent states. The transitions labelled by ξ_1 and ξ_2 are joined by the minimum. Afterwards, there does not exist the possibility of adding the following transition (in this case $\min(\psi_1, \psi_2)$) due to the fact that this would produce a change in the *stochastic* choice at the root. Besides, the states located after ρ_1 and ρ_2 are not equivalent, so they cannot be joined. Finally, we can add ψ_3 and $\min(\rho_3, \rho_4)$ as in s_{10} and s_{11}.

In order to keep the presentation as simple as possible, we have preferred to introduce our semantics on a class of (stochastic) labelled transition systems. As we said before, we will separate between *usual* actions and *stochastic* actions. Random variables associated with stochastic transitions may have any kind of probability distribution. In Section 2 we present this class of labelled transition systems. In Section 3, we define three notions of bisimulation (strong, weak, and weak stochastic). As usually, weak bisimulation is weaker than strong bisimulation. We will also show that the new defined weak stochastic bisimulation is weaker than weak bisimulation. Finally, in Section 4 we present our conclusions and some lines for future work.

2 Stochastic Labelled Transition Systems

In this section we define our model of stochastic processes. We consider a class of labelled transition systems where transitions are labelled either by an action or by a random variable. First, we introduce some concepts on random variables.

We will consider that the sample space (that is, the domain of random variables) is the set of real numbers $\mathbb{R}$ and that random variables take positive values only in $\mathbb{R}^+$, that is, given a random variable ξ we have $\mathrm{P}(\xi \leq t) = 0$ for all $t < 0$. The reason for this restriction is that random variables are always associated with time distributions. We also introduce the function $\oplus$. This operator will be used when random variables associated with the same state are combined.

Definition 1. Let ξ be a random variable. We define its *probability distribution function*, $F_\xi : \mathbb{R} \rightarrow [0,1]$, as the function such that $F_\xi(x) = \mathrm{P}(\xi \leq x)$, where $\mathrm{P}(\xi \leq x)$ is the probability that ξ assumes values less than or equal to x. We denote by *unit* the random variable such that $F_{unit}(x) = 1$ for all $x \geq 0$.

Let ξ_1, ξ_2 be independent random variables with probability distribution functions F_{ξ_1} and F_{ξ_2}, respectively. We define the *combined addition* of ξ_1 and ξ_2, denoted by $\xi_1 \oplus \xi_2$, as the random variable with probability distribution function $F_{\xi_1 \oplus \xi_2}(x) = F_{\xi_1}(x) + F_{\xi_2}(x) - F_{\xi_1}(x) \cdot F_{\xi_2}(x)$. This operator can be generalized to an arbitrary (finite) number of random variables. Let $\Psi = \{\xi_i\}_{i \in I}$ be a non-empty finite set of independent random variables. We define the *combined addition* of the variables in Ψ, denoted by $\oplus\Psi$, as the random variable such that $F_{\oplus\Psi}(x) = \sum_{\emptyset \subset \Phi \subseteq \Psi} (-1)^{(|\Phi|+1)} F_{\otimes\Phi}(x)$ where $F_{\otimes\Psi}(x) = \prod_{i \in I} F_{\xi_i}(x)$. □

Let us note that for singleton sets, $\Psi = \{\xi\}$, we have $\oplus\Psi = \xi$. Also note that this operator does not correspond to the usual definition of addition of random variables (we will denote the addition of random variables by $+$). Actually, it can be shown that $\oplus$ computes the minimum of a set of random variables.

Lemma 1. Let $\Psi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ be a non-empty set of independent random variables, and let ξ be a random variable distributed as the random variable $\min\{\xi_1, \xi_2, \ldots, \xi_n\}$. For all $x \in \mathbb{R}$ we have $F_\xi(x) = F_{\oplus\Psi}(x)$.

We suppose a fixed set of actions Act ($a, a', \ldots$ to range over Act) and a special action $\tau \notin$ Act to represent internal activity. We denote by Act_τ the set $\mathrm{Act} \cup \{\tau\}$ ($\alpha, \alpha', \ldots$ to range over Act_τ). We denote by $\mathcal{V}$ the set of random variables ($\xi, \xi', \psi, \ldots$ to range over $\mathcal{V}$). Finally, $\gamma, \gamma', \ldots$ will denote generic elements in $\mathrm{Act}_\tau \cup \mathcal{V}$.

Definition 2. A *stochastic labelled transition system* P is a pair $(S, \rightarrow)$ where S is a finite *set of states*, and $\rightarrow \subseteq S \times (\mathrm{Act}_\tau \cup \mathcal{V}) \times S$ is a *transition relation*.

We will use the following conventions: $s \xrightarrow{\gamma} s'$ stands for $(s, \gamma, s') \in \rightarrow$; $s \xrightarrow{\gamma}$ stands for there exists $s' \in S$ such that $s \xrightarrow{\gamma} s'$; we write $s \stackrel{\gamma}{\not\longrightarrow}$ if there does not exist such an $s' \in S$. We say that s is *stable* if $s \stackrel{\tau}{\not\longrightarrow}$. We denote by $\stackrel{\tau}{\Longrightarrow}$ the reflexive and transitive closure of $\xrightarrow{\tau}$; given $\gamma \neq \tau$, we write $s \stackrel{\gamma}{\Longrightarrow} s'$ if there exist s_1, s_2 such that $s \stackrel{\tau}{\Longrightarrow} s_1 \xrightarrow{\gamma} s_2 \stackrel{\tau}{\Longrightarrow} s'$. Given a set $A \subseteq \mathrm{Act}_\tau \cup \mathcal{V}$, we write $s \xrightarrow{A}$ (resp. $s \stackrel{A}{\Longrightarrow}$) if there exists $\gamma \in A$ such that $s \xrightarrow{\gamma}$ (resp. $s \stackrel{\gamma}{\Longrightarrow}$), and we write $s \stackrel{A}{\not\longrightarrow}$ (resp. $s \stackrel{A}{\not\Longrightarrow}$) if there does not exist such a γ. □

Intuitively, a transition $s \xrightarrow{a} s'$ indicates that a process may evolve from s to s' by performing the (visible) action a. A transition $s \xrightarrow{\tau} s'$ indicates that a process may internally evolve from s to s'. Finally, a transition $s \xrightarrow{\xi} s'$ expresses that a process may evolve from the state s to the state s' once the (random) delay indicated by ξ has been elapsed. In order to avoid *side effects*, we suppose that all the random variables labeling transitions of a process are independent. In particular, this implies that the same random variable cannot appear in different transitions. Note that this restriction does not forbid to have identically distributed random variables (as far as they have different names). This assumption would not be necessary if we defined stochastic delays by using probability distribution functions. Anyway, for the sake of convenience, we will sometimes use the same random variable in different transitions. For example, two transitions labelled by the same random variable ξ (e.g. s_9 in Figure 1) is a shorthand to indicate that these transitions are labelled by two identically distributed independent random variables ψ_1 and ψ_2.

The following examples show how the *race policy* (i.e. the fastest delay is always taken in a process) identifies some processes that *apparently* should not be equivalent.

Example 1. Consider s_{13} and s_{14} in Figure 1. Suppose that ξ is uniformly distributed over $[0, 2]$, ψ_1 is uniformly distributed over $[1, 5]$, and that the probability distribution function of ψ_2 is given by:

$$F_{\psi_2}(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \frac{x-1}{4} & \text{if } 1 < x \leq 4 \\ \frac{x}{2} - \frac{5}{4} & \text{if } 4 < x \leq \frac{9}{2} \\ 1 & \text{if } \frac{9}{2} < x \end{cases}$$

Taking into account that ψ_1 and ψ_2 are not identically distributed, we could think that s_{13} and s_{14} are not equivalent. However, as the fastest delay will be taken in these states, after two units of time ξ will be chosen with probability 1 because $\mathrm{P}(\xi \leq 2) = 1$. Since we have that ψ_1 and ψ_2 are identically distributed over the interval $[0, 4]$, so are they over $[0, 2]$. Thus, s_{13} and s_{14} should be equivalent. □

Example 2. Consider again s_{13} and s_{14} in Figure 1, but now suppose that ξ is uniformly distributed over $[0, \frac{1}{2}]$ and that ψ_1 and ψ_2 are defined as in Example 1. We have that the probability of performing either ψ_1 or ψ_2 would be zero. In this case, both s_{13} and s_{14} should be equivalent, and they should be also equivalent to s_2 (also in Figure 1). This is so because the delay associated with ξ is performed with probability 1 after a half of a unit of time, and at that time ψ_1 and ψ_2 cannot be performed. □

We also consider that internal actions are *urgent*, that is, given the fact that they do not need to interact with the environment, they will be performed as soon as possible, so no delay is allowed. This property, usually known as *maximal progress*, appears in most timed and stochastic models. Moreover, we will consider that if a state can evolve neither internally nor stochastically, this state can *wait* any amount of time before any visible action is performed.

Definition 3. Let $P = (S, \longrightarrow)$ be a stochastic labelled transition system, $s, s' \in S$, $\xi, \psi \in \mathcal{V}$, $t_0 \in \mathbb{R}^+$, and $C \subseteq S$. We define the following concepts:

Maximum waiting time. The *maximum waiting time* for a state s, denoted by $maxW(s)$, is defined as

$$maxW(s) = \begin{cases} 0 & \text{if } s \xrightarrow{\tau} \\ \min\{t \mid \exists \xi \in \mathcal{V} \colon s \xrightarrow{\xi} \wedge F_\xi(t) = 1\} & \text{otherwise} \end{cases}$$

Identically distributed random variables. We say that ξ and ψ are *identically distributed with respect to* states s and s', denoted by $\xi \asymp_{s,s'} \psi$, if $maxW(s) = maxW(s')$ and for all $t \leq maxW(s)$ we have $F_\xi(t) = F_\psi(t)$.

Feasible random variable. We say that the random variable ξ is *feasible with respect to* t_0, denoted by $fact_{t_0}(\xi)$, if $t_0 > \min\{t \mid F_\xi(t) > 0\}$.

Set positively reached. We say that s *may positively reach* C, denoted by $s \nearrow C$, if there exist $s' \in C$ and $\xi \in \mathcal{V}$ such that $s \xrightarrow{\xi} s'$ and $fact_{maxW(s)}(\xi)$. □

The function $maxW(s)$ will ensure maximal progress. If the state can evolve internally, it cannot wait to perform any delay, and so $maxW(s) = 0$. If the state is stable, then the function $maxW(s)$ computes the maximum amount of time that s can wait until a delay is performed with probability 1. We have considered $\min \emptyset = \infty$. So, if s is stable and has no stochastic transitions then it can wait as long as one of its transitions is enabled.

The predicate $\asymp_{s,s'}$ will be used to compare random variables. As previous examples illustrate, probability distribution functions associated with random variables are compared only in the interval of time limited by the maximum waiting time of the corresponding states. For example, if we consider Example 1 we have $\psi_1 \asymp_{s_{13},s_{14}} \psi_2$ because $maxW(s_{13}) = maxW(s_{14}) = 2$. For that reason, the states s_{13} and s_{14} should be equivalent. Meanwhile, in Example 2 the transitions ψ_1 and ψ_2 will not be taken into consideration (and so, we will say that they are not feasible).

In order to avoid the computation of the minimum of an empty set of random variables we have defined the predicate $s \nearrow C$. This predicate holds if s can reach $C \subseteq S$ by performing a stochastic transition.

3 Stochastic Bisimulations

Next we define our notions of bisimulation. The first two ones correspond to the typical notions of strong and weak bisimulations for stochastic processes (see for example [Her98,BG98] for similar notions on a Markovian setting). The third notion, that we call *weak stochastic bisimulation*, represents a refinement (i.e. it is *weaker*) of the previous notions. We consider that not only internal actions may be (partially) abstracted but also (in some cases) sequences of random variables may be abstracted.

We start by giving the definition of strong bisimulation. First, we introduce an auxiliary definition to combine random variables that, starting from the same state, reach equivalent states.

Definition 4. Let $P = (S, \longrightarrow)$ be a stochastic labelled transition system, and $C \subseteq S$. We define the *strong random variable* associated with the execution of stochastic transitions from a state $s \in S$ leading to C, denoted by $\xi_s(s, C)$, as $\xi_s(s, C) = \oplus\{\xi \mid \exists s' \in C : s \xrightarrow{\xi} s'\}$. □

The function $\xi_s(s, C)$ combines all the random variables leading from s to a state belonging to C by using the function $\oplus$. Let us remind that $\oplus$ computes the minimum of a set of random variables.

Definition 5. We say that an equivalence relation $\mathcal{R}$ is a *strong bisimulation* on $P = (S, \longrightarrow)$ if for all pair of states $s_1, s_2 \in S$ we have that $s_1 \mathcal{R} s_2$ implies:

- For all $\alpha \in \mathrm{Act}_\tau$, $s_1 \xrightarrow{\alpha} s_1'$ implies $\exists s_2'$ such that $s_2 \xrightarrow{\alpha} s_2'$ and $s_1' \mathcal{R} s_2'$.
- For all $C \in S/\mathcal{R}$, $s_1 \nearrow C$ implies $s_2 \nearrow C$ and $\xi_s(s_1, C) \asymp_{s_1,s_2} \xi_s(s_2, C)$.

We say that two states $s_1, s_2 \in S$ are *strongly bisimilar*, denoted by $s_1 \sim_s s_2$, if there exists a strong bisimulation that contains the pair (s_1, s_2). □

The definition of strong bisimulation mimics the one for non-stochastic systems: A transition from a state must be simulated by the same transition in the other one. The first clause of the definition is the usual one for strong bisimulation. The second clause is used to join several stochastic transitions leading to the same equivalence class. Note that $\xi_s(s_1, C)$ and $\xi_s(s_2, C)$ do not need to be equal for any time value; they have to take equal values for any time lower than or equal to the maximum waiting time of both states. Finally, let us note that the symmetric cases, where s_1 and s_2 exchange roles, are omitted because we already force $\mathcal{R}$ to be an equivalence relation (and so $\mathcal{R}$ is symmetric).

Example 3. Consider the states s_2, s_{13}, and s_{14} from Figure 1 together with the random variables defined in Examples 1 and 2. Let us define the sets of states C_1 and C_2 such that $s_2', s_{13}', s_{14}' \in C_1$ and $s_{13}'', s_{14}'' \in C_2$. Considering the values of the random variables given in Example 1, we have $\xi_s(s_{13}, C_i) \asymp_{s_{13},s_{14}} \xi_s(s_{14}, C_i)$ for $i \in \{1, 2\}$ (note that $maxW(s_{13}) = 2 = maxW(s_{14})$). So, $s_{13} \sim_s s_{14}$.

Let us consider the random variables as defined in Example 2. In this case we have that $\xi_s(s_2, C_1) \asymp_{s_2,s_{13}} \xi_s(s_{13}, C_1) \asymp_{s_{13},s_{14}} \xi_s(s_{14}, C_1)$. Note that the values $\xi_s(s_j, C_2)$ for $j \in \{2, 13, 14\}$ should not be computed. So, $s_2 \sim_s s_{13} \sim_s s_{14}$. □

Next we present a notion of weak bisimulation for stochastic processes. As in the strong case, we will also impose the condition of reachability (given in Definition 3) before we compute ξ_s. Nevertheless, the corresponding transitions have to reach C^τ instead of C, where $C^\tau = \{s \mid \exists s' \in C : s \stackrel{\tau}{\Longrightarrow} s'\}$.

Definition 6. We say that an equivalence relation $\mathcal{R}$ is a *weak bisimulation* on a stochastic labelled transition system $P = (S, \longrightarrow)$ if for any pair of states $s_1, s_2 \in S$ we have that $s_1 \mathcal{R} s_2$ implies:

- For all $\alpha \in \mathrm{Act}_\tau$, $s_1 \xrightarrow{\alpha} s_1'$ implies $\exists s_2'$ such that $s_2 \stackrel{\alpha}{\Longrightarrow} s_2'$ and $s_1' \mathcal{R} s_2'$.
- For all $C \in S/\mathcal{R}$, $s_1 \stackrel{\tau}{\Longrightarrow} s_1' \nearrow C^\tau$ implies $\exists s_2'$ such that $s_2 \stackrel{\tau}{\Longrightarrow} s_2' \nearrow C^\tau$ and $\xi_s(s_1', C^\tau) \asymp_{s_1',s_2'} \xi_s(s_2', C^\tau)$.

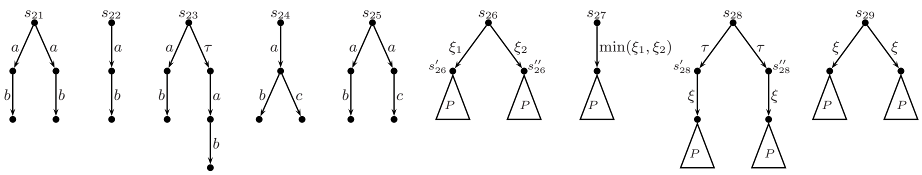

Fig. 2. Examples of strongly/weakly bisimilar processes.

We say that two states $s_1, s_2 \in S$ are *weakly bisimilar*, denoted by $s_1 \sim_w s_2$, if there exists a weak bisimulation that contains the pair (s_1, s_2). □

As in the definition of strong bisimulation, we distinguish between stochastic transitions and usual action transitions. The first clause of the previous definition is the usual one for weak bisimulation. The last clause checks whether the random variables associated with the states are identically distributed up to the corresponding maximum waiting time. In this case, in order to (partially) abstract internal transitions, it is also allowed the evolution by a (possibly empty) sequence of internal actions.

Strong and weak bisimulations fulfill the usual properties. The proof is an adaptation of the one given in [Mil89] by taking into account that, also in our setting, the relation $\sim_s$ (resp. $\sim_w$) can be defined as the union of all the relations that are strong (resp. weak) bisimulations.

Lemma 2. Let $P = (S, \longrightarrow)$ be a stochastic labelled transition system. The strong (resp. weak) bisimilarity relation on P, that is, $\sim_s$ (resp. $\sim_w$) is an equivalence relation on P. Moreover, $\sim_s$ (resp. $\sim_w$) is a strong (resp. weak) bisimulation on P and it is the largest strong (resp. weak) bisimulation on P.

Example 4. Consider Figure 2. States s_{21} and s_{22} are both strong and weak bisimilar. Besides, s_{23} is weakly equivalent (but not strongly) to both s_{21} and s_{22}. In contrast, s_{24} and s_{25} are not equivalent under any of the bisimulations defined in this paper. These examples show that these notions are conservative with respect to the non-stochastic framework (see forthcoming Lemma 3).

Stochastic transitions leading to the same equivalence class must be *joined*. Consider s_{26} and s_{27}. These two states are (strongly and weakly) bisimilar due to the fact that both stochastic transitions outgoing from s_{26} lead to the same equivalence class. In this way, the delay after which P is enabled will be the minimum of both random variables, that is, the same as in s_{27}. Consider now s_2 in Figure 1 and s_{28} in Figure 2. They are not strongly bisimilar because $s_2 \stackrel{\tau}{\not\longrightarrow}$ while $s_{28} \stackrel{\tau}{\longrightarrow}$. However, $s_2 \sim_w s_{28}$. The states s_{28} and s_{29} are not identified. If s_{28} evolves by performing a τ transition then s_{29} cannot simulate this transition. This is a desired result (and standard in stochastic models) because the identification of these two states would produce that a pure internal decision (as in s_{28}) should be treated as a *probabilistic* one. □

These two notions of bisimulation restricted either to labelled transition systems, or to stochastic labelled transition systems where distributions are always exponential, are equivalent to the classical notions of bisimulation, and to the notions of bisimulation defined in [Her98], respectively. In the latter case it is enough to take into account that when restricted to exponential distributions we have that, for all $s \in S$, $maxW(s)$ is equal either to 0 or ∞. Besides, for any $C \subseteq S$, we have that $s \nearrow C$ can be simplified as $\exists s' \in C$ such that $s \xrightarrow{\xi} s'$. Unfortunately, it is not easy to compare our definitions with the corresponding to models with general distributions. For example, [BG02] uses a preselection policy (instead of a race policy), while the clocks mechanism of [DKB98] is difficult to compare with our model.

Lemma 3. Restricted to non-stochastic labelled transitions systems, $\sim_s$ (resp. $\sim_w$) is equivalent to the strong (resp. weak) bisimulation defined in [Mil89].

Restricted to Markovian stochastic labelled transitions systems, $\sim_s$ (resp. $\sim_w$) is equivalent to the strong (resp. weak) bisimulation defined in [Her98].

Nevertheless, this notion of weak bisimulation presents a drawback. Specifically, it considers stochastic transitions almost as visible ones. Obviously, stochastic transitions cannot be considered to be just τ transitions because they carry some information that τ transitions do not. However, as discussed in the introduction, in some situations sequences of stochastic transitions should be identified. This is the case for s_1 and s_2 in Figure 1 for some particular random variables ξ_1, ξ_2, and ξ.

Next, we define our new notion of bisimulation that we call *weak stochastic bisimulation*. It is needed again an auxiliary function to compute the *probability* of reaching a set of states from a state. First, we introduce some predicates to indicate conditions on the *continuations* after evolving by internal and/or stochastic transitions.

Definition 7. Let $P = (S, \longrightarrow)$ be a stochastic labelled transition system, $\mathcal{R}$ be an equivalence relation on S, and $s \in S$. We define the following predicates:

Equivalence class. We denote by $[s]_\mathcal{R}$ the *equivalence class* induced by $\mathcal{R}$ on S containing s, that is, the set $[s]_\mathcal{R} = \{s' \in S \mid s\mathcal{R}s'\}$.

Stabilization of a state. We say that s' is a *stabilization of* s with respect to $\mathcal{R}$, denoted by $Stab_\mathcal{R}(s, s')$, if $s' \in [s]_\mathcal{R}$ and $s' \stackrel{\tau}{\not\longrightarrow}$, and for all $r \in S$ we have $s \stackrel{\tau}{\Longrightarrow} r$ implies $r \in [s]_\mathcal{R}$.

Stochastically nondeterministic state. We say that s is *stochastically nondeterministic* with respect to $\mathcal{R}$, and we write $Snd_\mathcal{R}(s)$, if there exist $s_1, s_2 \in S$ such that $s \nearrow [s_1]_\mathcal{R}$, $s \nearrow [s_2]_\mathcal{R}$, and $[s_1]_\mathcal{R} \neq [s_2]_\mathcal{R}$.

Deterministically reached state. We say that s *deterministically reaches* s' by *stochastic transitions* with respect to $\mathcal{R}$, denoted by $\mathcal{D}_\mathcal{R}(s, s')$, if $[s]_\mathcal{R} \neq [s']_\mathcal{R}$ and exists $s'' \in S$ such that $Stab_\mathcal{R}(s, s'')$ and $s'' \nearrow [s']_\mathcal{R}$, and for all $r \in S$ such that $s'' \nearrow [r]_\mathcal{R}$ we have $[s']_\mathcal{R} = [r]_\mathcal{R}$.

Stochastically deterministic state. We say that s is *stochastically deterministic* with respect to $\mathcal{R}$, denoted by $Sd_\mathcal{R}(s)$, if for any state $r \in S$ we have $s \stackrel{\tau}{\Longrightarrow} r$ implies $r \in [s]_\mathcal{R}$ and one of the following two conditions holds:

– $s \overset{\mathcal{V}}{\not\longrightarrow}$.
– For all $s' \in [s]_{\mathcal{R}}$, $\exists s'' \in S$ such that $s' \nearrow [s'']_{\mathcal{R}}$ implies $[s'']_{\mathcal{R}} = [s]_{\mathcal{R}}$.

□

The predicate $Stab_{\mathcal{R}}(s, s')$ indicates that either s is stable or it stays in its equivalence class after performing any sequence of internal transitions. We will use any stable state s' belonging to the class of s to study its associated random variables.

The predicate $Snd_{\mathcal{R}}(s)$ indicates the case when *continuations* after the performance of a stochastic action should not be added. We reject to (immediately) study the continuations if the states reached after performing stochastic transitions do not belong to the same equivalence class. As we have already say, it is important to note that not all the stochastic transitions outgoing from a state are considered. For example, the state s_{13} appearing in Example 2 is not stochastically nondeterministic. Consider s_{31} in Figure 4. In this case, the continuations after the execution of the initial stochastic transitions, (that is, after ξ_1, ξ_2 and ξ_3) are not equivalent if ψ_1 and ψ_2 are not identically distributed. That is, s'_{31} and s'''_{31} will not be equivalent. Let us consider that ξ_1, ξ_2 and ξ_3 are such that $maxW(s_{31}) > \min\{t \mid F_{\xi_i}(t) > 0\}$ for $i \in \{1, 2, 3\}$. In this case, all the stochastic transitions are considered. So, we have that s_{31} is a stochastically nondeterminist state.

Meanwhile, the predicate $\mathcal{D}_{\mathcal{R}}(s, s')$ represents the reverse case. The continuations will be joined when all the stochastic transitions outgoing from a state lead to states belonging to the same equivalence class. For example, in Figure 4, we have that s'_{37} deterministically reaches the state s''_{37}. So, if $\xi = \psi + (\psi_1 \oplus \psi_2)$ then s_{37} should be equivalent to s_2 in Figure 1.

Finally, the predicate $Sd_{\mathcal{R}}(s)$ represents the case when all the stochastic and internal transitions outgoing either from s or from any state of its equivalence class reach states belonging to $[s]_{\mathcal{R}}$. For example, in Figure 1, we have that s_5 is a stochastically deterministic state.

The following function represents the core of the definition of weak stochastic bisimulation. The key point is the addition of the continuations only when it does not distort the choice in the upper level. A similar approach, that is, to introduce an auxiliary function to define weak bisimulation, is taken in [BH97] although for a much simpler model.

Definition 8. Let $P = (S, \rightarrow)$ be a stochastic labelled transition system, $\mathcal{R}$ an equivalence relation on S, and $C \in S/\mathcal{R}$. We define the *weak stochastic random variable* associated with the stochastic transitions outgoing from a state s_0 of P leading to a state belonging to C, denoted by $\xi_{wst}(s_0, C, \mathcal{R})$, in Figure 3. □

Let us note that the previous function is partial because not all the classes of a given equivalence relation will be stochastically reached with a weak stochastic random variable. Let us remind that *unit* is the random variable given in Definition 1 and that for any class A, A^τ is the set $\{s \in S \mid \exists s' \in A : s \overset{\tau}{\Longrightarrow} s'\}$. The first clause is applied if s_0 belongs to the class we are interested in (or this class can be reached after the execution of several internal transitions) and it is

$$
\begin{cases}
\mathit{unit} & \text{if } s_0 \in C^\tau \ \wedge \ \mathit{Sd}_{\mathcal{R}}(s_0) \\
\xi_s(s, C^\tau) & \text{if } s_0 \notin C^\tau \ \wedge \ \mathit{Stab}_{\mathcal{R}}(s_0, s) \ \wedge \\
& \left(\begin{array}{l} s \xrightarrow{\text{Act}} \ \vee \ \mathit{Snd}_{\mathcal{R}}(s) \ \vee \\ \exists s' \in S : \left(\begin{array}{l} \mathcal{D}_{\mathcal{R}}(s, s') \ \wedge \\ \left(\begin{array}{l} s' \overset{\text{Act}}{\Longrightarrow} \ \vee \\ \exists s'' \in S : \ (\neg \mathit{Stab}_{\mathcal{R}}(s', s'') \ \vee \ \mathit{Snd}_{\mathcal{R}}(s'')) \end{array} \right) \end{array} \right) \end{array} \right) \\
\begin{array}{c} \xi_s(s, [s']^\tau_{\mathcal{R}}) \\ + \\ \xi_{wst}(s', C, \mathcal{R}) \end{array} & \begin{array}{l} \text{if } s_0 \notin C^\tau \ \wedge \ \mathit{Stab}_{\mathcal{R}}(s_0, s) \ \wedge \ s \not\xrightarrow{\text{Act}} \ \wedge \mathcal{D}_{\mathcal{R}}(s, s') \ \wedge \ s' \not\overset{\text{Act}}{\Longrightarrow} \ \wedge \\ \exists s'' \in S \colon \mathcal{D}_{\mathcal{R}}(s', s'') \end{array}
\end{cases}
$$

Fig. 3. Definition of the weak stochastic random variable $\xi_{wst}(s_0, C, \mathcal{R})$.

stochastically deterministic. For example, for s_4 and s_5 in Figure 1, if we consider $\mathcal{R}$ such that $[s_4]_{\mathcal{R}} = [s_5]_{\mathcal{R}}$ then the random variables associated with s_4 to reach $[s_4]_{\mathcal{R}}$ and with s_5 to reach $[s_5]_{\mathcal{R}}$ are identically distributed as *unit*.

The other two cases are considered when s_0 is not in C^τ. The second case is applied if there exists a stable state $s \in S$, reached from s_0 after performing internal transitions, and either s may perform a visible action, or not all of the feasible initial stochastic transition (or their continuations) lead to the same equivalence class. In this case, we do not *add* the continuations. Consider Figure 4 and an equivalence relation $\mathcal{R}$ such that $[s_{31}]_{\mathcal{R}} = \{s_{31}\}$, $[s'_{31}]_{\mathcal{R}} = \{s'_{31}, s''_{31}\}$, and $[s'''_{31}] = \{s'''_{31}\}$. We have $\xi_{wst}(s_{31}, [s'_{31}]_{\mathcal{R}}, \mathcal{R}) = \xi_{wst}(s_{31}, [s''_{31}]_{\mathcal{R}}, \mathcal{R}) = \xi_1 \oplus \xi_2$, and $\xi_{wst}(s_{31}, [s'''_{31}]_{\mathcal{R}}, \mathcal{R}) = \xi_3$.

The third clause is taken if all the feasible initial stochastic transitions are joined and their continuations lead to equivalent states. In this case, we should not stop *counting* after the initial transitions. We join the initial stochastic transitions (by using ξ_s) and then we add the results to the random variable corresponding to the continuations. Consider the states s_{15} and s_{17} in Figure 1 and an equivalence relation $\mathcal{R}$ such that $[s_{15}]_{\mathcal{R}} = \{s_{15}, s_{17}\} = [s_{17}]_{\mathcal{R}}$, $[s'_{15}]_{\mathcal{R}} = \{s'_{15}, s'_{17}\} = [s'_{17}]_{\mathcal{R}}$, and $[s'''_{15}]_{\mathcal{R}} = \{s'''_{15}, s^{iv}_{15}, s''_{17}\} = [s''_{17}]_{\mathcal{R}}$. In this case, $\xi_{wst}(s'_{15}, [s'''_{15}]_{\mathcal{R}}, \mathcal{R}) = \xi_{wst}(s'_{17}, [s''_{17}]_{\mathcal{R}}, \mathcal{R}) = \psi_3 + (\rho_3 \oplus \rho_4)$. Finally, it is obvious to check that the three cases are mutually exclusive.

In the previous definitions of bisimulation we have imposed $s \nearrow C$, before computing $\xi_s(s, C)$, to ensure that this function was defined. But this condition is not enough to assure that $\xi_{wst}(s, C, \mathcal{R})$ is defined. There exist two reasons why this function could be indefinite: Either there does not exist a path to reach C from s or there exists a path but it is *illegal*. We will say that a path $p \equiv s \overset{\xi_1}{\Longrightarrow} s_1 \ldots s_{n-1} \overset{\xi_n}{\Longrightarrow} s_n$ is *legal* if the states of the path fulfill the conditions of any of the three clauses of the definition of $\xi_{wst}(s, C, \mathcal{R})$. Intuitively, we need to consider only those paths such that the function ξ_{wst} can be applied in every step of the computation.

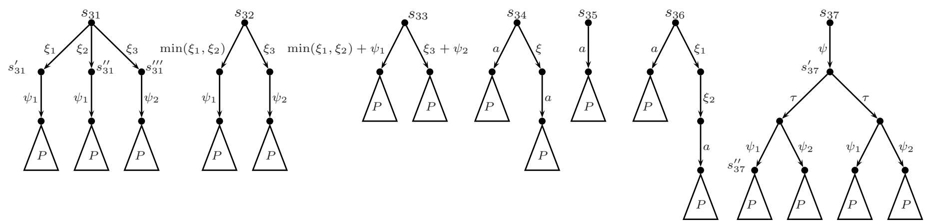

Fig. 4. Examples of weakly stochastic bisimilar processes.

Definition 9. Let $P = (S, \rightarrow)$ be a stochastic labelled transition system, $C \subseteq S$, $\mathcal{R}$ an equivalence relation on S, and $s_0, s_i \in S, \xi_i \in \mathcal{V}$ with $1 \leq i \leq n$. We say that $p \equiv s_0 \overset{\xi_1}{\Longrightarrow} s_1 \overset{\xi_2}{\Longrightarrow} s_2 \cdots s_{n-1} \overset{\xi_n}{\Longrightarrow} s_n$ is a *legal path to reach* C with respect to $\mathcal{R}$, denoted by $legal_{\mathcal{R}}(p, C)$, if one of the following conditions hold:

1. $s_0 \in C^\tau$, $Sd_{\mathcal{R}}(s_0)$, and $n = 0$.
2. $s_0 \notin C^\tau$, $n = 1$, there exists $s \in S$ such that $Stab_{\mathcal{R}}(s_0, s)$, $fact_{maxW(s)}(\xi_1)$, $s_1 \in C^\tau$, and $\left(\begin{array}{l} s \xrightarrow{\text{Act}} \ \vee\ Snd_{\mathcal{R}}(s) \ \vee\ (\mathcal{D}_{\mathcal{R}}(s, s_1) \ \wedge\ s_1 \overset{\text{Act}}{\Longrightarrow}) \ \vee \\ (\mathcal{D}_{\mathcal{R}}(s, s_1) \ \wedge\ \forall s'' \in S: \ Stab_{\mathcal{R}}(s_1, s'') \text{ implies } Snd_{\mathcal{R}}(s'')) \end{array} \right)$.
3. $s_0 \notin C^\tau$, there exists $s \in S$ such that $Stab_{\mathcal{R}}(s_0, s)$, $s \overset{\text{Act}}{\not\longrightarrow}$, $\mathcal{D}_{\mathcal{R}}(s, s_1)$, and for all $1 \leq i \leq n-1$ we have $s_i \overset{\text{Act}}{\not\Longrightarrow}$ and $\mathcal{D}_{\mathcal{R}}(s_i, s_{i+1})$ and, finally, $s_n \in C^\tau$ and $Sd_{\mathcal{R}}(s_n)$.

We say that C is *stochastically reached* from s_0 with respect to $\mathcal{R}$, denoted by $s_0 \Nearrow_{\mathcal{R}} C$, if there exists a path p from s_0 such that $legal_{\mathcal{R}}(p, C)$. □

Let us remark that if there exists a legal path p from s to C then all the paths from s to states in C whose stochastic transitions are feasible are also legal. This is so because the condition to be a legal path considers all the stochastic transitions that each state can perform. Actually, the predicate $s \Nearrow_{\mathcal{R}} C$ is used to check that one of the clauses in Definition 8 holds.

Lemma 4. Let $P = (S, \rightarrow)$ be a stochastic labelled transition system, $C \subseteq S$, $\mathcal{R}$ an equivalence relation in S and $s \in S$. If $s \Nearrow_{\mathcal{R}} C$ then the function $\xi_{wst}(s, C, \mathcal{R})$ is well defined.

Once we have defined the previous predicate, we present the notion of weak stochastic bisimulation.

Definition 10. We say that an equivalence relation $\mathcal{R}$ is a *weak stochastic bisimulation* on $P = (S, \rightarrow)$ if for any pair of states $s_1, s_2 \in S$ we have that $s_1 \mathcal{R} s_2$ implies:

– For all $\alpha \in \text{Act}_\tau, s_1 \xrightarrow{\alpha} s'_1$ implies $\exists s'_2$ such that $s_2 \overset{\alpha}{\Longrightarrow} s'_2$ and $s'_1 \mathcal{R} s'_2$.
– For all $C \in S/\mathcal{R}$, $s_1 \overset{\tau}{\Longrightarrow} s'_1$, $s'_1 \overset{\tau}{\not\longrightarrow}$, and $s'_1 \Nearrow_{\mathcal{R}} C$ implies $\exists s'_2$ such that $s_2 \overset{\tau}{\Longrightarrow} s'_2 \Nearrow_{\mathcal{R}} C$ and $\xi_{wst}(s'_1, C, \mathcal{R}) \asymp_{s'_1, s'_2} \xi_{wst}(s'_2, C, \mathcal{R})$.

We say that two states $s_1, s_2 \in S$ are *weakly stochastic bisimilar*, denoted by $s_1 \sim_{wst} s_2$, if there exists a weak stochastic bisimulation that contains the pair (s_1, s_2). $\square$

The difference between $\sim_w$ and $\sim_{wst}$ comes from the definition of ξ_{wst} and the new condition $(s \nearrow\!\!\!\nearrow_{\mathcal{R}} C)$ to ensure that $\xi_{wst}(s, C, \mathcal{R})$ is computed only if there exists a legal path from s reaching C.

Example 5. Consider s_{31}, s_{32}, and s_{33} in Figure 4. Let us suppose that all the stochastic transitions are feasible with respect to the state they lead from. We have that s_{31} and s_{32} are both weakly and weakly stochastic bisimilar. On the contrary, s_{33} is not equivalent to them. The point is that while the choice appearing in s_{31} and s_{32} is made between ξ_3 and the minimum of ξ_1 and ξ_2, in the case of s_{33} the choice is *distorted* by ψ_1 and ψ_2.

Consider s_{34}, s_{35}, and s_{36}, the first two ones are weakly stochastic bisimilar (they are not weakly bisimilar). However, s_{34} and s_{36} are not weakly stochastic bisimilar because s_{34} is always able to perform a while this is not the case for s_{36}. If the stochastic transition $s_{36} \xrightarrow{\xi_1}$ is performed then this process will not be able to perform a until the delay ξ_2 is elapsed. $\square$

This new bisimulation also fulfills the usual properties. The proof is a little bit more involved than the ones for strong and weak bisimulations because of the definition of ξ_{wst}.

Lemma 5. Let $P = (S, \rightarrow)$ be a stochastic labelled transition system. Weak stochastic bisimilarity on P (that is, the relation $\sim_{wst}$) is an equivalence relation on P. Moreover, $\sim_{wst}$ is a weak stochastic bisimulation on P and the largest weak stochastic bisimulation on P.

Theorem 1. Let $P = (S, \rightarrow)$ be a stochastic labelled transition system, and $s_1, s_2 \in S$. We have that $s_1 \sim_s s_2$, implies $s_1 \sim_w s_2$. Besides, if $s_1 \sim_w s_2$, then $s_1 \sim_{wst} s_2$.

Corollary 1. $\sim_s \subsetneq \sim_w \subsetneq \sim_{wst}$.

4 Conclusions and Future Work

In this paper we have studied bisimulation semantics for a class of stochastic processes with general distributions. We have presented a notion of weak stochastic bisimulation to solve some of the counterintuitive results that appeared in previous definitions. One key point that we have not addressed in this paper is how our results can be extended to a more complex language. In particular, to define a parallel operator in the context of general distributions is always involved. We obtained some preliminary results in [LNR04], but the formulation of the resulting (weak) bisimulation relation is very involved.

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References

[ABC+94] M. Ajmone Marsan, A. Bianco, L. Ciminiera, R. Sisto, and A. Valenzano. A LOTOS extension for the performance analysis of distributed systems. *IEEE/ACM Transactions on Networking*, 2(2):151–165, 1994.

[BC00] M. Bernardo and W.R. Cleaveland. A theory of testing for markovian processes. In *CONCUR'2000, LNCS 1877*, pages 305–319. Springer, 2000.

[BG98] M. Bernardo and R. Gorrieri. A tutorial on EMPA: A theory of concurrent processes with nondeterminism, priorities, probabilities and time. *Theoretical Computer Science*, 202:1–54, 1998.

[BG02] M. Bravetti and R. Gorrieri. The theory of interactive generalized semi-Markov processes. *Theoretical Computer Science*, 282(1):5–32, 2002.

[BH97] C. Baier and H. Hermanns. Weak bisimulation for fully probabilistic processes. In *Computer Aided Verification'97, LNCS 1254*, pages 119–130. Springer, 1997.

[BW90] J.C.M. Baeten and W.P. Weijland. *Process Algebra*. Cambridge Tracts in Computer Science 18. Cambridge University Press, 1990.

[DKB98] P.R. D'Argenio, J.-P. Katoen, and E. Brinksma. An algebraic approach to the specification of stochastic systems. In *Programming Concepts and Methods*, pages 126–147. Chapman & Hall, 1998.

[GHR93] N. Götz, U. Herzog, and M. Rettelbach. Multiprocessor and distributed system design: The integration of functional specification and performance analysis using stochastic process algebras. In *16th Int. Symp. on Computer Performance Modelling, Measurement and Evaluation (PERFORMANCE'93), LNCS 729*, pages 121–146. Springer, 1993.

[Hen88] M. Hennessy. *Algebraic Theory of Processes*. MIT Press, 1988.

[Her98] H. Hermanns. *Interactive Markov Chains*. PhD thesis, Universität Erlangen-Nürnberg, 1998.

[HHK02] H. Hermanns, U. Herzog, and J.-P. Katoen. Process algebra for performance evaluation. *Theoretical Computer Science*, 274(1-2):43–87, 2002.

[Hil96] J. Hillston. *A Compositional Approach to Performance Modelling*. Cambridge University Press, 1996.

[Hoa85] C.A.R. Hoare. *Communicating Sequential Processes*. Prentice Hall, 1985.

[HS00] P.G. Harrison and B. Strulo. SPADES – a process algebra for discrete event simulation. *Journal of Logic Computation*, 10(1):3–42, 2000.

[LN00] N. López and M. Núñez. NMSPA: A non-markovian model for stochastic processes. In *International Workshop on Distributed System Validation and Verification (DSVV'2000)*, pages 33–40, 2000.

[LN01] N. López and M. Núñez. A testing theory for generally distributed stochastic processes. In *CONCUR 2001, LNCS 2154*, pages 321–335. Springer, 2001.

[LNR04] N. López, M. Núñez, and F. Rubio. An integrated framework for the analysis of asynchronous communicating stochastic processes. *Formal Aspects of Computing*, 16(3):238–262, 2004.

[Mil89] R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.